1a. Consider $f(p) = \langle p, a\rangle \quad \forall p \in S$

Since S is compact, $\exists\, q \in S$ such that $f(q) = \max_{p\in S} \langle p, a\rangle$

Let $v \in T_p(S)$, $\exists\, \alpha : (-\varepsilon, \varepsilon) \to S$ such that $\alpha(0) = q$, $\alpha'(0) = v$

Consider $f\circ\alpha(t) = \langle \alpha(t), a\rangle$ which is maximum at $t=0$

$0 = f\circ\alpha'(0) = \langle \alpha'(0), a\rangle = \langle v, a\rangle$

So $a \perp T_q S$, a is parallel to normal line at q

1b. We want $f(p) = \langle p, a\rangle$ is constant $\forall p \in S$

Sufficient to show $\langle \alpha(t), a\rangle$ is constant for any smooth curve α on S

$f\circ\alpha(t) = \langle \alpha(t), a\rangle$

$f\circ\alpha'(t) = \langle \alpha'(t), a\rangle$

$= 0$

So f is constant on α

Since S is connected, $f(p)$ is constant $\forall p \in S$

2a. Let $f: S_1 \to S_2$ be a diffeomorphism between surfaces

$(\Rightarrow)$ Suppose S_1 is orientable

then $\exists$ collection of chart $\{X_\alpha : U_\alpha \to S_1\}$

such that $S_1 = \bigcup_\alpha X_\alpha(U_\alpha)$

$\varphi_{\alpha\beta} = X_\beta^{-1} \circ X_\alpha$ is orientation preserving diffeomorphism

(i.e. det of Jacobian matrix of $\varphi_{\alpha\beta} > 0$)

Consider $Y_\alpha = f\circ X_\alpha : U_\alpha \to S_2$

then $\bigcup_\alpha Y_\alpha(U_\alpha) = S_2$

$Y_\beta^{-1} \circ Y_\alpha = (f\circ X_\beta)^{-1} \circ (f\circ X_\alpha)$

$= X_\beta^{-1} \circ f^{-1} \circ f \circ X_\alpha$

$= X_\beta^{-1} \circ X_\alpha$ is orientation preserving diffeomorphism

For the converse, same argument

2b. Consider the chart $X(u,v) = \left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{u^2+v^2-1}{u^2+v^2+1} \right)$

$$X : \mathbb{R}^2 \longrightarrow S^2 \setminus \{(0,0,1)\}$$

$$X^{-1}(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z} \right)$$

Let $Y = f \circ X(u,v) = \left(-\frac{2u}{u^2+v^2+1}, -\frac{2v}{u^2+v^2+1}, -\frac{u^2+v^2-1}{u^2+v^2+1} \right)$

Then $X^{-1} \circ Y : \mathbb{R}^2 \setminus \{(0,0)\} \longrightarrow \mathbb{R}^2 \setminus \{(0,0)\}$

$$X^{-1} \circ Y(u,v) = \left(-\frac{u}{u^2+v^2}, -\frac{v}{u^2+v^2} \right)$$

$$\begin{vmatrix} | & | \\ \frac{\partial X^{-1}\circ Y}{\partial u} & \frac{\partial X^{-1}\circ Y}{\partial v} \\ | & | \end{vmatrix} = \begin{vmatrix} -\frac{1}{u^2+v^2} + \frac{2u^2}{(u^2+v^2)^2} & \frac{2uv}{(u^2+v^2)^2} \\ \frac{2uv}{(u^2+v^2)^2} & -\frac{1}{u^2+v^2} + \frac{2v^2}{(u^2+v^2)^2} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{u^2-v^2}{(u^2+v^2)^2} & \frac{2uv}{(u^2+v^2)^2} \\ \frac{2uv}{(u^2+v^2)^2} & \frac{v^2-u^2}{(u^2+v^2)^2} \end{vmatrix}$$

$$= -\frac{1}{(u^2+v^2)^2} < 0$$

Since S^2 is connected, f is orientation reversing

3. Let $p \in S_1$, $\exists$ nbd U of p such that $f : U \to f(U)$ is diffeomorphism

Since S_2 is orientable, $\exists$ collection of chart $\{ X_\alpha : W_\alpha \subseteq \mathbb{R}^2 \to S_2 \}$ such that $\bigcup_\alpha X_\alpha(W_\alpha) = S_2$

$X_\beta^{-1} \circ X_\alpha$ is orientation preserving

Then $\exists$ one chart $\{ X_i : W_i \to S_2 \}$ such that $f(p) \in X_\alpha(W_\alpha)$

Let $U_p = X_i^{-1}(f(U) \cap W_\alpha)$

$V_p = f^{-1}(X_i(U_i)) \subseteq U$

$Y_p = f^{-1} \circ X_i : U_p \longrightarrow V_p$

Then $\bigcup_p \{ Y_p : U_p \to S_1 \}$ is collection of chart

Suppose $Y_q = f^{-1} \circ X_\beta$

$Y_q^{-1} \circ Y_p = X_\beta^{-1} \circ X_i$ is orientation preserving

4. Let $X(u,v) : U \subseteq \mathbb{R}^2 \longrightarrow V \subseteq S$, $\bar{X}(\bar{u},\bar{v}) : \bar{U} \subseteq \mathbb{R}^2 \longrightarrow \bar{V} \subseteq S$

Let $p \in V \cap \bar{V}$, $\varphi = \bar{X}^{-1} \circ X : X^{-1}(V \cap \bar{V}) \longrightarrow \bar{X}^{-1}(V \cap \bar{V})$

Let $\varphi(u,v) = (\bar{u}(u,v), \bar{v}(u,v))$.

Let $v \in T_p S$, $\exists\ \alpha : (-\varepsilon, \varepsilon) \longrightarrow V \cap \bar{V} \subseteq S$ such that $\alpha(0) = p$, $\alpha'(0) = v$

Then $X(u(t), v(t)) = \alpha(t) = \bar{X}(\bar{u}(t), \bar{v}(t))$

$$u'(0) \frac{\partial X}{\partial u} + v'(0) \frac{\partial X}{\partial v} = \alpha'(0) = \bar{u}'(0) \frac{\partial \bar{X}}{\partial \bar{u}} + \bar{v}'(0) \frac{\partial \bar{X}}{\partial \bar{v}}$$

$$\bar{u}'(0) = \frac{d}{dt} \bar{u}(t) \Big|_{t=0} = \frac{d}{dt} \bar{u}(u(t), v(t)) \Big|_{t=0}$$

$$= u'(0) \frac{\partial \bar{u}}{\partial u} + v'(0) \frac{\partial \bar{u}}{\partial v}$$

$$\bar{v}'(0) = u'(0) \frac{\partial \bar{v}}{\partial u} + v'(0) \frac{\partial \bar{v}}{\partial v}$$

$$u'(0) \frac{\partial X}{\partial u} + v'(0) \frac{\partial X}{\partial v} = \left(u'(0) \frac{\partial \bar{u}}{\partial u} + v'(0) \frac{\partial \bar{u}}{\partial v} \right) \frac{\partial \bar{X}}{\partial \bar{u}} + \left(u'(0) \frac{\partial \bar{v}}{\partial u} + v'(0) \frac{\partial \bar{v}}{\partial v} \right) \frac{\partial \bar{X}}{\partial \bar{v}}$$

Hence $$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial \bar{u}}{\partial u} & \frac{\partial \bar{u}}{\partial v} \\ \frac{\partial \bar{v}}{\partial u} & \frac{\partial \bar{v}}{\partial v} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$